Kinetic Equations

Text of the Exercises

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Exercise 1

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be bounded with $\nabla_z b \in L^\infty\left([0,T] \times \mathbb{R}^d; M_d(\mathbb{R})\right)$.

Recall that given a measure μ on $\mathbb{R}^d$ and a measurable function φ we use the following notation:

$$\mu(\varphi) := \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x). \tag{1}$$

Consider the *Liouville equation* given by

$$\begin{cases} \partial_t \mu_t(\psi) = \mu_t\left(b(t,\cdot) \cdot \nabla_z \psi\right), & \forall t \in [0,T],\ \forall \psi \in C_c^\infty\left(\mathbb{R}^d\right), \\ \mu_t(\psi)|_{t=0} = \mu_0(\psi), & \forall \psi \in C_c^\infty\left(\mathbb{R}^d\right). \end{cases} \tag{2}$$

Suppose that $M = \{\mu_t |\ t \in [0,T]\}$ is a family of measures such that for each $t \in [0,T]$ the measure μ_t is $\mathcal{L}^d$ absolutely continuous (we will write equivalently $\mathcal{L}^d$-a.c., where $\mathcal{L}^d$ is the Lebesgue measure in d dimensions); moreover, assume that exists $f_M \in C^1\left([0,T] \times \mathbb{R}^d\right)$ such that $d\mu_t(z) = f_M(t,z)\, dz$.

Under these assumptions, prove that M is a solution to (2) if and only if f_M is a classical solution to

$$\begin{cases} \partial_t f + \operatorname{div}_z(bf) = 0, \\ f(t,z)|_{t=0} = f_M(0,z), \quad \forall z \in \mathbb{R}^d. \end{cases} \tag{3}$$

Remark. *Notice that if $d = 6$, $z = (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $b(t,x,v) = (v, E(t,x))$ the Liouville equation is written as*

$$\begin{cases} \partial_t \mu_t(\psi) = \mu_t\left(v \cdot \nabla_x \psi + E(t,x) \cdot \nabla_v \psi\right), & \forall t \in [0,T],\ \forall \psi \in C_c^\infty\left(\mathbb{R}^d\right), \\ \mu_t(\psi)|_{t=0} = \mu_0(\psi), & \forall \psi \in C_c^\infty\left(\mathbb{R}^d\right). \end{cases} \tag{4}$$

Exercise 2

Recall that if $p \in \mathbb{N}$, we defined in class

$$\mathcal{P}_p\left(\mathbb{R}^d\right) := \left\{\mu \text{ Borel measure} |\ \mu \geqslant 0,\ \mu\left(\mathbb{R}^d\right) = 1, \int_{\mathbb{R}^d} |x|^p\, d\mu(x) < +\infty\right\}. \tag{5}$$

Consider now a sequence of measures $\{\mu_k\}_{k \in \mathbb{N}} \subseteq \mathcal{P}_1\left(\mathbb{R}^d\right)$ and $\mu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$. We say that μ_k *converges to μ weakly* and we write $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ if

$$\lim_{k \to +\infty} \int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x), \qquad \forall \varphi \in C_b\left(\mathbb{R}^d\right), \tag{6}$$

where $C_{\mathrm{b}}\left(\mathbb{R}^d\right)$ denotes the space of bounded continuous functions.

Prove that the following properties are equivalent:

(i) $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ and

$$\lim_{k\to+\infty} \int_{\mathbb{R}^d} |x|\, d\mu_k(x) = \int_{\mathbb{R}^d} |x|\, d\mu(x)\,; \tag{7}$$

(ii) $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ and

$$\limsup_{k\to+\infty} \int_{\mathbb{R}^d} |x|\, d\mu_k(x) \leqslant \int_{\mathbb{R}^d} |x|\, d\mu(x)\,; \tag{8}$$

(iii) $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ and

$$\lim_{R\to+\infty} \limsup_{k\to+\infty} \int_{|x|\geqslant R} |x|\, d\mu_k(x) = 0; \tag{9}$$

(iv) For any $\varphi \in C\left(\mathbb{R}^d\right)$ such that there exists a positive constant C with $|\varphi(x)| \leqslant C(1+|x|)$ for all $x \in \mathbb{R}^d$ we have

$$\lim_{k\to+\infty} \int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x)\,. \tag{10}$$

Exercise 3

Recall that the space $\mathrm{Lip}\left(\mathbb{R}^d\right)$ is defined as the set of function φ such that $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} < +\infty$, where

$$\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} := \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{d_1(x,y)}, \tag{11}$$

and where $d_1(\cdot,\cdot)$ is the euclidean distance between two points.

Prove that the function $\mathcal{W}_1 : \mathcal{P}_1\left(\mathbb{R}^d\right) \times \mathcal{P}_1\left(\mathbb{R}^d\right) \to \mathbb{R}$ defined for all $\mu,\ \nu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$ as

$$\mathcal{W}_1(\mu,\nu) := \sup\left\{\mu(\varphi) - \nu(\varphi) \mid \varphi \in \mathrm{Lip}\left(\mathbb{R}^d\right),\ \|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} \leqslant 1\right\}. \tag{12}$$

defines a distance on $\mathcal{P}_1\left(\mathbb{R}^d\right)$.